Math 138 – Practice midterm

Due date: October 15, 2025

Instructions:

- You have 2 hours to complete this exam.
- No external resources are allowed.
- Upload your completed practice midterm to Crowdmark.

Word of advice: While not 100%, we can likely tell if you use AI on this exam. Ultimately the grade on the practice midterm is *completion based*. To cheat on this will likely not result in a higher score on the practice midterm, but will almost certainly result in a lower score on the actual exam. Similarly, the two hour limit is not enforceable, but you should still follow it as it will help you get a better sense of your preparedness for the actual exam.

Question 1. (15 pts)

Spot the error in the following proof, and explain why it is wrong.

Proposition. *Every function* $f: \{1, \ldots, n\} \to \{1, \ldots, n\}$ *has a fixed point (i.e., for some* x *one has* $f(x) = x$*).*

Proof. We prove this by induction on n. When $n = 1$ the only function $f: \{1\} \to \{1\}$ satisfies $f(1) = 1$, so has a fixed point. For the induciton hypothesis, assume the claim is true for any function $g: \{1, \ldots, n\} \to \{1, \ldots, n\}$, and consider a function

$$f: \{1 \ldots, n+1\} \to \{1, \ldots, n+1\}.$$

Case 1: If $f(n+1) = n+1$ we're done.

Case 2: If $f(n+1) \neq n+1$, we can consider the function $g: \{1, \ldots, n\} \to \{1, \ldots, n\}$ given by $f(i) = g(i)$. By induction hypothesis there is some $x \in \{1, \ldots, n\}$ such that $x = g(x) = f(x)$.

Thus, in either case we see f has a fixed point as desired. □

Solution: The error is in the induction hypothesis, specifically in Case 2. Namely, we defined g to be a function from $\{1, \ldots, n\}$ to itself via the rule $g(i) = f(i)$ for $i \in \{1, \ldots, n\}$. But, this is not necessarily well-defined as $g(i)$ could be $n + 1$ for some $i \in \{1, \ldots, n\}$. So, the best we may do is define g as a function $\{1, \ldots, n\} \to \{1, \ldots, n+1\}$, but then the induction hypothesis does not apply to g.

Rubric:

- **(2 pts)** Explaining why the result is wrong.
- **(3 pts)** Identifying that the issue is with the induction hypothesis step.
- **(5 pts)** Explicating the exact point that is an issue.
- **(5 pts)** Giving a counterexample showing why this point is not correct.

Question 2. (15 pts)

Let $n \geqslant k \geqslant 0$ be integers. Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Solution 1: We prove this by algebraic manipulation. Namely, it was established in class that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

In particular, we aim to show that

$$\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}. \tag{1}$$

But, we may rewrite the terms on the right-hand side as

$$\frac{(n-1)!}{k!(n-k-1)!} = \frac{(n-1)!(n-k)}{k!(n-k)!}$$

and

$$\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!k}{k!(n-k)!}.$$

Thus, using these expressions, and multiplying through (1) by $k!(n-k)!$ shows our desired equation is equivalent to showing that

$$n! = (n-1)!(n-k) + (n-1)!k.$$

But, dividing both sides by $(n-1)!$ this reduces to showing that

$$n = (n-k) + k,$$

which is true. ■

Solution: Observe that the left-hand side is counting the number of ways of selecting k numbers from a set of $\{1, \ldots, n\}$ elements. This is also what the right-hand side counts. Indeed, $\binom{n-1}{k}$ counts the number of ways of choosing the k things from $\{1, \ldots, n\}$ where none of them are n (i.e., the number of ways of choosing k things from $\{1, \ldots, n-1\}$). But, $\binom{n-1}{k-1}$ then counts the number of ways of choosing k things from $\{1, \ldots, n\}$ where one of those k things in n, as this amounts to exactly the number of ways choosing the remaining $k-1$ things from $\{1, \ldots, n-1\}$. Thus, the sum of these accounts exactly for the number of ways of choosing k things from $\{1, \ldots, n\}$ as desired. ■

Rubric:

- **(5 pts)** Coherence in proof writing.
- **(5 pts)** Valid strategy (e.g., algebraic manipulation or counting).
- **(5 pts)** Executed strategy without errors.

Question 3. (20 pts)

Suppose that $x \in \mathbb{Q}$. Use the p-adic valuation to prove that if $x^{\frac{2}{3}}$ belongs to $\mathbb{Q}$ then $x^{\frac{1}{3}}$ belongs to $\mathbb{Q}$.

Solution: By the result stated in class, $x^{\frac{1}{3}}$ belongs to $\mathbb{Q}$ if and only if $\operatorname{sgn}(x) = \pm 1$ has a third root in $\mathbb{Q}$, and for all primes p one has $3 \mid v_p(x)$. But, $\operatorname{sgn}(x) = \pm 1$ always has a third root in $\mathbb{Q}$—$\sqrt[3]{1} = 1$ an $\sqrt[3]{-1} = -1$. So, it suffices to prove that $3 \mid v_p(x)$ for all primes p. But, by this same theorem we know that as $x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}}$ is in $\mathbb{Q}$, that $3 \mid v_p(x^2)$ for all primes p. But, $v_p(x^2) = 2v_p(x)$ as proven in class, so $3 \mid 2v_p(x)$. As 2 and 3 are coprime this implies that $3 \mid v_p(x)$ for all primes p as desired. ■

Rubric:

- **(5 pts)** Coherence in proof writing.
- **(3 pts)** Correctly defining/using the p-adic valuation.
- **(5 pts)** Pursuing a correct strategy

  (e.g. Recalling that $x^{\frac{1}{3}}$ is in $\mathbb{Q}$ if and only if $3 \mid v_p(x)$ for all p, and $\operatorname{sgn}(x)$ has a cube root.
- **(2 pts)** Correct identifying at some point that 2 and 3 being coprime was important.
- **(5 pts)** Correctly executing strategy.

Question 4. 20 pts

Let A and B be subsets of a set S. Prove that

$$A \triangle B = (A \cup B) \cap (A^c \cup B^c).$$

Solution: Recall that $A \triangle B = (A - B) \cup (B - A)$, i.e., those x such that x is in A or B but not both.

Suppose first that $x \in A \triangle B$. If $x \in A$ then $x \notin B$. So, if $x \in A$ then $x \in B^c$, and so $x \in A \cup B$ and $x \in A^c \cup B^c$, so $x \in (A \cup B) \cap (A^c \cup B^c)$. The same argument applies by symmetry if, instead, $x \in B$. Moreover, as x is in A or B this accounts for all cases. Thus, we see that $A \triangle B \subseteq (A \cup B) \cap (A^c \cup B^c)$.

Conversely, suppose that $x \in (A \cup B) \cap (A^c \cup B^c)$. Then, x belongs to A or B, and also x belongs to A^c or B^c. If x belongs to A, then as it belongs also to one of A^c or B^c, and it can't belong to A^c by definition, it belongs to B^c. So, in that case $x \in A \cap B^c = A - B \subseteq A \triangle B$. By symmetry, if x belongs to B then x belongs to $A \triangle B$. So, $(A \cup B) \cap (A^c \cup B^c) \subseteq A \triangle B$.

As $A \triangle B \subseteq (A \cup B) \cap (A^c \cup B^c)$ and $(A \cup B) \cap (A^c \cup B^c) \subseteq A \triangle B$, we deduce that $A \triangle B = (A \cup B) \cap (A^c \cup B^c)$, as desired. ■

Rubric:

- **(5 pts)** Coherence in proof writing.
- **(3 pts)** Correctly recalling/using the definition of $\triangle$
- **(2 pts)** Giving a correct strategy of approach (i.e., show that each side contains the other).
- **(5 pts)** Correctly arguing for left-hand side contained in right-hand side.
- **(5 pts)** Correctly arguing for right-hand side contained in left-hand side.

Question 5. (30 pts)

Prove by induction that if T_n is the number of length $n \geqslant 1$ strings in the symbols $0, 1, 2$ with exactly one 2 then

$$T_n = n \cdot 2^{n-1}$$

Solution: We proceed by induction.

Base case: When $n = 1$ we see that the only possible such strings in $\{0, 1, 2\}$ of length one containing exactly 2 are just the string 2 itself. So, $T_1 = 1 = 1 \cdot 2^{1-1}$.

Inductive hypothesis: Assume that $T_n = n \cdot 2^{n-1}$. Consider then a string of length $n + 1$ in the symbols $0, 1, 2$ with exactly one 2. We have two cases.

Case 1: The first digit is a 2. In this case the remaining n-length string can consist of any string in $\{0, 1\}$ of which there are 2^n each choices (for each of the n places there are two choices of 0 or 1).

Case 2: If the first digit is not a 2, then it is either a 0 or 1. In either case, the remaining length n string can be any string in $0, 1, 2$ with exactly one 2. By induction hypothesis this is $T_n = n \cdot 2^{n-1}$. Thus, the total number of cases here is $2 \cdot (n \cdot 2^{n-1}) = n \cdot 2^n$.

As cases 1 and 2 are disjoint and account for all possibilities we see that

$$T_{n+1} = 2^n + n \cdot 2^n = (n + 1)2^n,$$

as desired.

Rubric:

- **(5 pts)** Coherence in proof writing.
- **(5 pts)** Correctly identifying the correct base step(s).
- **(8 pts)** Giving correct general structure for inductive hypothesis argument (e.g., explicating where the usage of exactly one two is important).
- **(12 pts)** Correctly executing proof of inductive hypothesis.